Table 1 Modal weighting factors (a_i), selected locations (ξ_i), and objective function (f)

Case	a_1	a_2	a_3	a_4	ξ_1	ξ_2	f
A	1	1	1	1	0.2837	0.7143	0.7652
B	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	0.2985	0.7015	0.1712
C	1	$\frac{1}{4}$	$\frac{1}{9}$	0	0.3333	0.6667	0
D	1	4	1	1	0.0402	0.5050	1.0721

through contour lines. The selected values of ξ_i correspond to the local minima. In case C the minimization of the objective function leads to the trivial solution $\sigma_1[\boldsymbol{B}(\boldsymbol{\xi})] = 0$, i.e., $f = 0$. The feasible area of the design variables ξ_1 and ξ_2 is limited by the constraint (6a). The selected locations for cases A, B, C, and D are represented in Fig. 1.

The results from Fig. 1 (or Table 1) show that the optimal locations are sensitive to the assumed proportions among a_i.

In case C the selected positions are the nodal points of the third mode (the interfering one) which corresponds to $f = 0$. In fact, the definition of f in (5) shows that, if $m = 1$, its minimum corresponds to $\sigma_1[\boldsymbol{B}(\boldsymbol{\xi})] = 0$, i.e., the sensors are placed at points with zero amplitude in the $(n + 1)$th mode. This means that the result in case C is obtained regardless of the values of a_1, a_2, and a_3 (provided that they are not 0). Nevertheless, it is not guaranteed that the matrix A is well conditioned.

In case D the sensors are not symmetrically located because of the asymmetry in the deflection. However, local minima ($\xi_1 = 0.29$, $\xi_2 = 0.71$ and $\xi_1 = 0.46$, $\xi_2 = 0.54$) corresponding to symmetric solutions exist as well. A solution close to the first one would be chosen if $a_3 = a_4 = 0$.

Conclusions

This paper presents a new criterion for selecting the positions of concentrated sensors for the measurement of the modal components of distortions in flexible structures. The optimization criteria are the minimization of the condition number of a matrix related to the modal decomposition and of the perturbation caused by upper modes.

Initial results show that the obtained locations depend on the expected proportions among the modes. The problem is sensitive to the influence of the interfering ones.

If the number of interfering modes is taken as equal to 1, the optimization process is strongly affected by the minimization of the influence of that mode, and so the selected locations are its nodal points. In that case, the condition number of the main matrix is not guaranteed to be small. This drawback may be avoided by taking into account several interfering modes.

If no certainty about the dominant mode is available, several redundant sets of sensors (corresponding to different expected proportions among the modes) could be installed. Relative weighting among its measurements might be adapted on-line according to real-time observed proportions.

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Simplified Calculation of Eigenvector Derivatives with Repeated Eigenvalues

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Introduction

DERIVATIVES of eigenvalues and eigenvectors with respect to structural parameters play an important role in structural design, identification, and optimization, and so research in this area has drawn the attention of many scholars for a long time. An excellent survey paper by Adelman and Haftka[1] summarizes the progress in the study of the modal sensitivities through 1986.

In 1976, Nelson[2] presented an efficient algorithm for computing derivatives of eigenvectors for the general real eigensystems with nonrepeated eigenvalues. The greatest advantage of Nelson's method is that it preserves the symmetry and bandedness of the original eigensystem and requires the knowledge of only those eigenvectors that are to be differentiated. But it cannot directly deal with cases of repeated eigenvalues that often occur in many practical engineering structures, such as the wheelsets on train, rotor systems, and geometrically symmetric structures. Ojalvo,[3] Mills-Curran,[4] Dailey,[5] and Shaw and Jayasuriya[6] extended Nelson's method for solving the derivatives of eigenvalues and eigenvectors of the structure with repeated eigenvalues. Hou and Kenny[7] introduced an alternate eigenvector derivative matrix formulation and reparameterized the multivariable eigenproblem into an eigenproblem that is in terms of a single positive-valued design parameter to obtain the eigenvalue and eigenvector approximate analysis for repeated eigenvalue problems. Recently, Liu et al.[8] considered the contribution of the truncated modes to eigenvector derivatives and proposed a more accurate method to calculate the eigenvector derivatives.

Assuming that the repeated-root modal derivatives have the form

$$\frac{\partial X}{\partial p} = X' = V + XC$$

The focus of the extended Nelson's method[3–5] is concentrated on the problem of how to determine the unique eigenvectors X and its coefficient matrix C, not on the problem of how to determine the matrix V. For solving the matrix $V = [v_1 \mid v_2 \mid \cdots \mid v_m]$, Ojalvo[3] and Dailey[4] suggested that v_i can be obtained by appropriately eliminating some rows and columns of the system matrix $(K - \lambda M)$ and solving the resultant linear equation. Unfortunately, this method may fail in some circumstances.[9] A more rigorous approach has been presented by Mills-Curran.[4] But it is difficult to implement on computer.

This paper deals with this problem and proposes an efficient method to determine the first part of the repeated-root eigenvector derivative (e.g., the matrix V) by introducing a set of nonmodal vectors that are easily obtained from the modes associated with the repeated eigenvalue. Those vectors are orthogonal to the repeated-root eigenvectors and consist of a basis of N-dimensional space together with the known repeated-root eigenvectors, and so vectors v_i can be expressed as a linear combination of those nonmodal vectors. Consequently, the coefficients are determined. As a numerical example, a cantilever beam is presented to illustrate the application of the proposed procedure. Accuracy of the new method is examined by a finite difference scheme.

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Theoretical Background

The real, symmetric eigenproblem considered in this paper is

$$(K - \lambda M)x = 0 \tag{1}$$

with the following orthonormalization condition:

$$x_i^T M x_i = 1, \qquad i = 1, \ldots, N \tag{2}$$

$$x_i^T M x_j = 0, \qquad i \neq j; i, j = 1, \ldots, N \tag{3}$$

where K is the structural stiffness matrix, M the mass matrix, λ the eigenvalue, x_i the eigenvector corresponding to λ_i, and N the total degrees of freedom of the structure. Assume that the eigenvalue problem has m repeated eigenvalues, that is, $\lambda_1 = \lambda_2 = \cdots = \lambda_m$, the corresponding eigenvectors are denoted by $x_i, i = 1, 2, \ldots, m$. In this case, the computation for derivatives of the eigenvalues and eigenvectors is not straightforward, because the eigenvectors of the repeated eigenvalues are not unique. In fact, any linear combination of the eigenvectors also is an eigenvector of repeated eigenvalues.

Now let

$$X = [x_1 \mid x_2 \mid \cdots \mid x_m] \tag{4}$$

and

$$\bar{x}_i = X a_i, \qquad i = 1, 2, \ldots, m \tag{5}$$

then vector $\bar{x}_i$ also is an eigenvector of structure corresponding to the repeated eigenvalues λ_i.

For convenience, denote

$$F_i = K - \lambda_i M, \qquad i = 1, \ldots, m \tag{6}$$

Note that λ_i is m repeated eigenvalues, and we have

$$F_1 = F_2 = \cdots = F_m \tag{7}$$

and the eigenproblem corresponding to the case of repeated eigenvalues can be written as

$$F_i \bar{x}_i = 0, \qquad i = 1, \ldots, m \tag{8}$$

Differentiating Eq. (8) yields

$$F_i' \bar{x}_i + F_i \bar{x}_i' = 0, \qquad i = 1, \ldots, m \tag{9}$$

where the prime symbol ′ denotes the first derivative with respect to structural design parameter p.

Substituting from Eq. (5) and premultiplying by X^T give

$$X^T F_i' X a_i + X^T F_i \bar{x}_i' = 0, \qquad i = 1, 2, \ldots, m \tag{10}$$

Note that the columns of the matrix X are eigenvectors of the original eigenproblem, and Eq. (10) becomes

$$X^T F_i' X a_i = 0, \qquad i = 1, 2, \ldots, m \tag{11}$$

Expanding Eq. (11)

$$\left[X^T (K' - \lambda_i M') X - \lambda_i' X^T M X\right] a_i = 0, \qquad i = 1, \ldots, m \tag{12}$$

Using Eqs. (2) and (3), we get

$$\left[X^T (K' - \lambda_i M') X - \lambda_i' I\right] a_i = 0, \qquad i = 1, \ldots, m \tag{13}$$

where I is an $m \times m$ unit matrix. Equation (13) is a subeigenvalue problem of order m about the derivatives of the repeated eigenvalues and the coefficient vector a_i. From this equation the derivatives of the repeated eigenvalues can be computed by

$$\lambda_i' = x_i^T [K' - \lambda_i M'] x_i, \qquad i = 1, \ldots, m \tag{14}$$

In this paper, we assume that the derivatives of the repeated eigenvalues λ_i' and the coefficient vector a_i can be obtained uniquely from the subeigenvalue problem described by Eq. (13). By solving Eq. (13), we have a set of unique eigenvectors of the original eigenproblem corresponding to $\lambda_i, i = 1, \ldots, m$:

$$\bar{X} = XA \tag{15}$$

where $\bar{X} = [\bar{x}_1 \mid \bar{x}_2 \mid \cdots \mid \bar{x}_m]$ and $A = [a_1 \mid a_2 \mid \cdots \mid a_m]$ is an $m \times m$ unique orthogonal matrix.

Assume that the eigenvector derivatives $\bar{x}_i'$ have the following form:

$$\bar{x}_i' = v_i + \sum_{k=1}^{m} c_{ki} \bar{x}_k, \qquad i = 1, 2, \ldots, m \tag{16}$$

Many papers[4,5] have been published on the problem of how to determine the coefficient matrix $C = (c_{ki})$ in Eq. (16). Their formulas are excerpted here.

For diagonal terms of matrix C,

$$c_{ii} = -\bar{x}_i^T \left(\tfrac{1}{2} M' \bar{x}_i + M v_i\right), \qquad i = 1, 2, \ldots, m \tag{17}$$

and for off-diagonal terms,

$$c_{ki} = \frac{\bar{x}_k^T (K'' - 2.\lambda_i' M' - \lambda_i M'') \bar{x}_i + 2.\bar{x}_k^T F_i' v_i}{2 \cdot (\lambda_i' - \lambda_k')}$$
$$i \neq k; i, k = 1, 2, \ldots, m \tag{18}$$

This paper addresses the problem of how to solve the vectors v_i efficiently.

Substituting Eq. (16) into Eq. (9) and noting that $\bar{X}$ is the eigenvector matrix for the repeated eigenvalue, we have

$$F_i v_i = -F_i' \bar{x}_i, \qquad i = 1, 2, \ldots, m \tag{19}$$

Because F_i is of order N and its rank is $(N - m)$, it cannot be inverted, and so Eq. (19) cannot be solved directly. Ojalvo[3] and Dailey[5] have suggested solving v_i by eliminating appropriate m rows and m columns from F_i along with the corresponding rows from the right-hand side of Eq. (19). Unfortunately, their methods may fail in some circumstances, as pointed out in Refs. 4 and 9. Mills-Curran[4] proposed a method to partition F_i into the following form:

$$\begin{pmatrix} F_i^{AA} & F_i^{AB} \\ F_i^{BA} & F_i^{BB} \end{pmatrix} \tag{20}$$

so that F_i^{AA} is nonsingular, and then Eq. (19) is partitioned in the same manner as Eq. (20). This gives a linear equation for solving v_i as

$$F_i^{AA} v_i^A = (F_i' \bar{x}_i)^A, \qquad i = 1, \ldots, m \tag{21}$$

and

$$v_i = \begin{Bmatrix} v_i^A \\ 0 \end{Bmatrix}, \qquad i = 1, \ldots, m \tag{22}$$

However, the partition of Eq. (19) is difficult to implement on computer. An efficient and easy implementation procedure will be given in the following section for solving v_i.

New Method

Denoting the unknown $(N - m)$ eigenvectors of the original eigenproblem as

$$x_{m+1}, x_{m+2}, \ldots, x_N \tag{23}$$

Those eigenvectors span an $(N - m)$ dimensional subspace, denoted by span$[x_{m+1}, x_{m+2}, \ldots, x_N]$, and so vectors v_i belong to this subspace and are orthogonal to $\bar{x}_j$, that is,

$$\bar{x}_j^T M v_i = 0, \qquad j = 1, \ldots, m; \qquad i = m+1, m+2, \ldots, N - m \tag{24}$$

Table 1 Comparison of results computed by the new method (NM) and by the finite difference scheme (FDS)

Repeated eigenvalue	0.18414E+1		0.73481E+2		0.84056E+3		0.70809E + 4	
	NM	FDS	NM	FDS	NM	FDS	NM	FDS
Eigenvalue and eigenvector derivatives	0.09185E+0	0.09176E+0	0.33066E+2	0.33048E+2	0.46048E+3	0.46048E+3	0.67431E+4	0.67431E+4
	−0.20518E−2	−0.20499E−2	−0.78797E−2	−0.78772E−2	−0.32937E−2	−0.32951E−2	0.33466E−2	0.33430E−2
	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0
	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0
	0.51086E−2	0.51037E−2	0.35916E−1	0.35899E−1	−0.46713E−1	−0.46655E−1	−0.18862E−1	−0.18841E−1
	0.12002E−2	0.11990E−2	−0.12390E−3	−0.12352E−3	0.91225E−2	0.91170E−2	0.11392E−2	0.11380E−2
	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0
	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0	0.00000E+0
	−0.19989E−2	−0.19971E−2	−0.49395E−1	−0.49370E−1	−0.16812E−1	−0.16789E−1	0.17156E−1	0.17139E−1

The objective of this section is to construct a set of basis vectors of the subspace span $[x_{m+1}, x_{m+2}, \ldots, x_N]$, then to express vector v_i as a linear combination of those basis vectors, and finally to determine the coefficients in the linear combination.

To this end, solve the following homogeneous linear equation:

$$\bar{X}^T M y = 0 \tag{25}$$

Because the rank of the matrix $\bar{X}^T M$ is m, from Eq. (25) we can get $(N - m)$ solution vectors that are linear independent:

$$y_1, y_2, \ldots, y_{N-m} \tag{26}$$

Generally, the multiplicity of the repeated eigenvalue λ_i is very small compared with the total degrees of freedom of the structure, i.e., $m \ll N$, the computational work for solving y_j is also small. The variables $y_1, y_2, \ldots, y_{N-m}$ are basis vectors of the subspace span $[x_{m+1}, x_{m+2}, \ldots, x_N]$ and have the following property:

$$y_j^T F_i y_j \neq 0, \qquad j = 1, 2, \ldots, N - m \tag{27}$$

In fact, y_j belongs to the subspace span $[x_{m+1}, x_{m+2}, \ldots, x_N]$ and can be expressed as

$$y_j = \sum_{k=m+1}^{N} b_k x_k \tag{28}$$

where at least one of b_k is nonzero. Substituting Eq. (28) into the left-hand side of Eq. (27) and noting that x_k are eigenvectors of the original eigenproblem, we have

$$y_j^T F_i y_j = \sum_{k=m+1}^{N} b_k^2(\lambda_k - \lambda_i), \qquad j = 1, 2, \ldots, N - m \tag{29}$$

and so Eq. (29) is valid.

Now let us orthogonalize y_j with respect to the system matrix $F_i (F_1 = F_2 = \cdots = F_m)$ by using the following extended Schmidt orthogonalization process.

Let

$$\bar{y}_1 = y_1 \tag{30}$$

and

$$\bar{y}_j = \sum_{k=1}^{j-1} b_{jk}\bar{y}_k + y_j, \qquad j \geq 2 \tag{31}$$

where the coefficients b_{jk} are determined by

$$b_{jk} = -\frac{\bar{y}_k^T F_i y_j}{\bar{y}_k^T F_i \bar{y}_k}, \qquad k = 1, 2, \ldots, j - 1 \tag{32}$$

Then $\bar{y}_j$ has the following properties:

$$\bar{y}_j^T F_i \bar{y}_k = \delta_{jk}\bar{y}_j^T F_i \bar{y}_j, \qquad j, k = 1, 2, \ldots, N - m \qquad i = 1, 2, \ldots, m \tag{33}$$

where δ_{jk} is the Kronecker delta function.

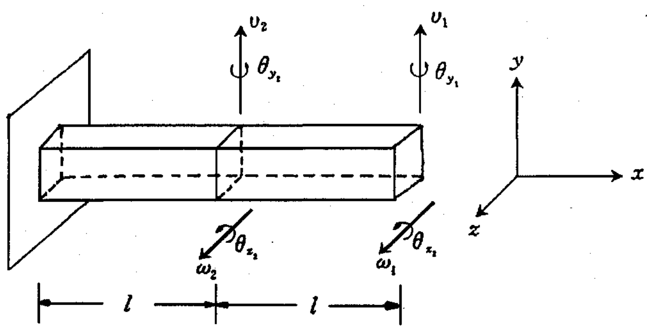

Fig. 1 Cantilever beam.

Because $y_1, y_2, \ldots, y_{N-m}$ are basis vectors of subspace span $[x_{m+1}, x_{m+2}, \ldots, x_N]$ and $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{N-m}$ are equivalent to $y_1, y_2, \ldots, y_{N-m}$, $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{N-m}$ are also basis vectors of the subspace span $[x_{m+1}, x_{m+2}, \ldots, x_N]$. Thus vector v_i can be expressed as a combination of $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{N-m}$:

$$v_i = \sum_{k=1}^{N-m} d_{ki}\bar{y}_k \tag{34}$$

the coefficients d_{ki} can be determined as follows.

Substituting Eq. (34) into Eq. (19) gives

$$\sum_{k=1}^{N-m} d_{ki} F_i \bar{y}_k = -F_i' \bar{x}_i, \qquad i = 1, 2, \ldots, m \tag{35}$$

Premultiplying Eq. (35) by $\bar{y}_k^T$ and using Eqs. (33) yield

$$d_{ki}\bar{y}_k^T F_i \bar{y}_k = -\bar{y}_k^T F_i' \bar{x}_i, \qquad i = 1, 2, \ldots, m \tag{36}$$

for $k = 1, 2, \ldots, N - m$. Thus

$$d_{ki} = -\frac{\bar{y}_k^T F_i' \bar{x}_i}{\bar{y}_k^T F_i \bar{y}_k}, \qquad i = 1, 2, \ldots, m; \qquad k = 1, 2, \ldots, N - m \tag{37}$$

This completes the solution for v_i. The procedure is summarized as follows:

1) Solve Eq. (25) and obtain the initial estimates of the vectors $y_1, y_2, \ldots, y_{N-m}$.

2) Orthogonalize $y_1, y_2, \ldots, y_{N-m}$ with respect to F_i ($F_1 = F_2 = \cdots = F_m$) by using Eqs. (30–32), and obtain vectors $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{N-m}$.

3) Compute d_{ki} by using Eq. (37) for $i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, N - m$.

4) Compute v_i by using Eq. (34), $i = 1, 2, \ldots, m$.

Numerical Example

The proposed procedure is illustrated by a square cantilever beam[5] (Fig. 1) that has four twice-repeated eigenvalues. The derivatives of the eigenvectors corresponding to the four twice-repeated eigenvalues are computed by the new method and the accuracy of those derivatives is tested by a finite difference scheme. The results are listed in Table 1.

Conclusions

An efficient and practical procedure based on the generalized Schmidt orthogonalization is presented to compute the eigenvector derivatives when repeated eigenvalues exist. The proposed procedure preserves the merits of the published methods but need not solve a high-order linear equation. Furthermore, the new method can be easily implemented on computer. As an example, we computed the eigenvector derivatives of repeated eigenvalues for a cantilever beam and compared the accuracy of the proposed method with a finite difference scheme.

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Effects of Geometries, Clearances, and Friction on the Composite Multipin Joints

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Introduction

PREDICTIONS of strength and failure mode for laminated composite plates with pin-loaded holes are important for structural engineers. The calculations require precise knowledge of the load distribution around the holes. Therefore, precise calculation of contact stresses around the hole, including consideration of unknown contact area, is important.

To date, many investigators have studied the pin-loaded plate problem analytically[1–3] and numerically.[4–10] In the literature, pin elasticity,[3,6] friction,[1–4,6,8] and contact clearance between pin and hole[1,3,4,6–9] have been studied as the important design factors. Also, in the contact area, it is generally accepted that there are two distinct regions: nonslip and slip. But, because of extreme difficulties, only a few papers[3,4,6,8] have included the nonslip region. Furthermore, although several workers[5,7,8,10] have studied the multipin-joint problem, most, except Kim and Kim,[8] considered the frictionless case or only the slip region in the frictional case because of the complexity.

In this Note, an extensive parametric study of composite multipin joints is performed by the penalty finite element method.[7,8] Design parameters, geometric factors, contact clearances, and friction are considered. Pin elasticity is not considered here since previous work[3] has shown that it is not as important as the other factors. To investigate anisotropic behavior of laminated composite plates, two representative quasi-isotropic laminates, $[0/\pm45/90\ \mathrm{deg}]_s$ and $[0_2/\pm45\ \mathrm{deg}]_s$, are used in the analysis. For the multipin-joint models, two pins in series and two in parallel are examined. Geometric factors include width of the plate (W), distance from upper edge of the plate to the hole center (E), and distance between the pins (G). Results include contact distributions for each case and the pin-loading analysis for two pins in series.

Description of the Problem

To analyze the stress around the holes in a pin-loaded anisotropic plate, the problem is idealized as a two-dimensional plane stress problem, where the pin is assumed to be rigid.

The pin-joint problem is naturally formulated as the frictional contact problem. A general class of contact problem with friction is characterized by the following set of equations and inequalities:

$$\begin{aligned} \sigma_{ij,j} + f_i &= 0 \quad \text{in} \quad \Omega \\ u_i &= 0 \quad \text{on} \quad \Gamma_D \\ \sigma_{ij} n_j &= t_i \quad \text{on} \quad \Gamma_F \end{aligned} \tag{1}$$

and on Γ_C
if $u_n - s < 0$

$$\sigma_n = 0, \qquad \sigma_T = 0 \tag{2}$$

if $u_n - s = 0$

$$\sigma_n < 0, \qquad u_T = 0 \qquad \text{if } |\sigma_T| < \mu|\sigma_n| \tag{3}$$

$$\sigma_n < 0, \qquad u_T = -\nu\sigma_T \qquad \text{if } |\sigma_T| = \mu|\sigma_n| \tag{4}$$

where Ω is a smooth domain and Γ_D is a displacement-prescribed boundary, Γ_F is a force-prescribed boundary, Γ_C is a candidate contact boundary, and μ is the friction coefficient. These systems of equations and inequalities describe a class of the signorini problems that obey the Coulomb friction law. As mentioned in the preceding section, the contact region can be divided into two distinct regions: the nonslip region is characterized in Eq. (3), and the slip region is characterized in Eq. (4).

Solution Procedure

To calculate contact stresses around the hole, the penalty finite element method[7,8] is used. For the calculation, the following three procedures are performed. First, the nonfrictional normal contact stress is calculated. In this stage, the extended interior penalty technique is used. For an iterative method, the standard Newton–Raphson scheme is used. Second, utilizing previous nonfrictional normal contact stress, the frictional contact problem with known normal contact stress is solved. In this stage, a regularization procedure for the friction term is performed and a successive tangent stiffness scheme is used for the iteration. Third, frictional force obtained in the second stage is added to total external force, and the equilibrium state is searched again. Until the global equilibrium state is reached, the previous three stages are repeated. For the global convergence criterion, relative error of normal stress is used. Details of the solution to the contact problem can be found in Refs. 7 and 8.

Numerical Results

Two representative cases of real multipin joints are analyzed: the plate with two holes in series and the plate with two holes in parallel. The two quasi-isotropic laminates considered here have stacking sequences of $[0/\pm45/90\ \mathrm{deg}]_s$ and $[0_2/\pm45\ \mathrm{deg}]_s$. The material

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